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M. T. NARANIENGAR, M.A.

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The Hon'ble R. P. PARANJPYE, M.A., D.Sc.,
Prof. A. C. L. WILKINSON, M.A., F.R.A.S.,
and others.

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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. It should be remembered that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with more familiar results

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

The Journal is open to contributions from members as well a[s] subscribers. The Editors may also accept contributions from others.

Contributors will be supplied, if so desired, with extra copies of the contributions at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be neatly and accurately drawn on separate slips.

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The roots of $\Delta(p) = 0$ are known* to correspond to purely imaginary values of ql; and they are expressed by the formula

$$\left.\begin{aligned} ql &= \lambda i, \quad p = -k\,\lambda^2/l^2 = \alpha, \\ \text{where} \quad \tan\lambda &= -(\lambda/hl), \end{aligned}\right\} \qquad \dots \qquad \dots (15)$$

and positive values only are to be used for λ.

Hence

$$\frac{1}{\alpha}\cdot\frac{F(\alpha)}{\Delta'(\alpha)} = -2\,\frac{\left(\dfrac{\lambda}{l}\cos\dfrac{\lambda\xi}{l} + h\sin\dfrac{\lambda\xi}{l}\right)\sin\left(\lambda - \dfrac{\lambda x}{l}\right)}{(1+hl)\cos\lambda - \lambda\sin\lambda}$$

$$= -2\,\frac{\left(\dfrac{\lambda}{l}\cos\dfrac{\lambda\xi}{l} + h\sin\dfrac{\lambda\xi}{l}\right)\left(\tan\lambda\cos\dfrac{\lambda x}{l} - \sin\dfrac{\lambda x}{l}\right)}{(1+hl) - \lambda\tan\lambda}$$

$$= -\frac{2}{l}\,\frac{\left(\lambda\cos\dfrac{\lambda\xi}{l} + hl\sin\dfrac{\lambda\xi}{l}\right)\left(\lambda\cos\dfrac{\lambda x}{l} + hl\sin\dfrac{\lambda x}{l}\right)}{hl(1+hl) + \lambda^2}$$

the final reduction following from (15).

Thus for the temperature in the rod, we have

$$V = \frac{2Q}{l}\sum \frac{\phi(\xi)\,.\,\phi(x)\,.\,e^{-k\frac{\lambda^2}{l^2}t}}{hl(1+hl)+\lambda^2}, \qquad \dots \qquad \dots (16)$$

where $\phi(x)$ denotes $\left(\lambda\cos\dfrac{\lambda x}{l} + hl\sin\dfrac{\lambda x}{l}\right)$ for the sake of simplicity.

7. *Semi-infinite rod. Source at $x = \xi$. Radiation at $x = 0$ into a medium at zero.*

The symbolical form of the solution in this case can be obtained by proceeding as before; or it can be deduced from the previous case by making $l \to \infty$ in (14) above. We thus get

$$V_2 = \frac{Q}{2}\left\{ e^{q\xi} + \frac{q-h}{q+h}\,e^{-q\xi}\right\} qe^{-qx}$$

$$= \frac{Q}{2}\left\{ qe^{-q(x-\xi)} + qe^{-q(x+\xi)} - \frac{2h}{h+q}\,qe^{-q(x+\xi)}\right\}. \quad (17)$$

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* Bromwich, *Proc. Camb. Phil. Soc.*, 20, p. 414.

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Now, since $$\frac{1}{h+q} = \int_0^\infty e^{-\lambda(h+q)}\,d\lambda,$$

we can write (17) in the form

$$V_2 = \frac{Q}{2}\Big\{ q e^{-q(x+\xi)} + q e^{-q(x+\xi)}$$

$$-2h\int_0^\infty q e^{-q(x+\xi)}\, e^{-\lambda(h+q)}\,d\lambda \Big\}$$

$$= \frac{Q}{2}\Big[q e^{-q(x-\xi)} + q e^{-q(x+\xi)}$$

$$-2h\int_0^\infty e^{-h\lambda}\, q e^{-q(x+\xi+\lambda)}\,d\lambda \Big]. \qquad (18)$$

To evaluate this, we appeal to the formula (5), *viz.*

$$q e^{-qz} = \frac{1}{\sqrt{\pi k t}}\, e^{-z^2/4kt};$$

so that the temperature at time t is given by

$$V = \frac{Q}{2\sqrt{\pi k t}}\Big\{ e^{-(x-\xi)^2/4kt} + e^{-(x+\xi)^2/4kt}$$

$$-2h\int^\infty e^{-h\lambda}\,.\,e^{-(x+\xi+\lambda)^2/4kt}\,d\lambda \Big\}.$$

Time, Space, Matter and Mind.*

BY S. V. RAMAMURTHI, I.C.S.

VII.

The fundamental equation of four-dimensional geometry is

$$r^3 = x^3 + y^3.$$

Let $y = 0$.

Then $r^3 = x^3$.

If $r = a,\ x^3 = a^3$

There are three values of x which make $x^3 = a^3$.

Let them be $+a,\ \wedge a,\ \barwedge a$;

(plus a, dinus a, trinus a).

(Dinus a and trinus a are not the imaginary cube roots of unity, but are real in 4dimensional space, just as minus a.)

Now

$$+ a \wedge a \barwedge a = 0,$$

$$(\wedge a)(\barwedge a) = +1,$$

$$\begin{aligned}(x+a)(x \wedge a)(x \barwedge a) &= (x+a)(x^2 \wedge ax \barwedge ax + a^2)\\ &= (x+a)(x^2 - ax + a^2)\\ &= x^3 + a^3.\end{aligned}$$

$$\therefore \quad \begin{aligned}x^3 + y^3 &= (x+y)(x \wedge y)(x \barwedge y)\\ &= (y+x)(y \wedge x)(y \barwedge x).\end{aligned}$$

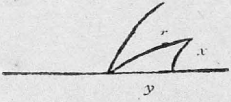
If we put $rE^{\theta} = y + x$, where θ is the angle between y and r and E is a constant,

then $$rE^{\barwedge\theta} = y \wedge x.$$

and $$rE^{\barwedge\theta} = y \barwedge x.$$

$$\therefore \quad r^3E^{\theta \wedge \theta \barwedge \theta} = y^3 + x^3.$$

$$\therefore \quad r^3 = y^3 + x^3.$$



* Sections I to VI of this paper were published in J.I.M.S., Vol. XV, No. 1. (Feb. 1923).

Hence, consistently with the relation

$$r^3 = x^3 + y^3,$$

we can take $$rE^\theta = x + y$$

where E is a constant.

$E^\circ = 1$, $E^p = 1$, where p is a right angle in this space.

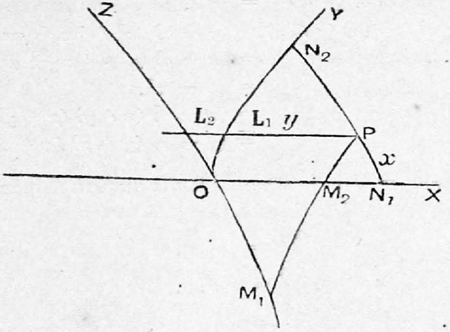
Every straight line in this plane has three directions—*positive, dinitive, trinitive* (as we may call them).

Every line in this plane is real (unlike in three-dimensional space), where the y line gives imaginary lengths through the equation

$$re^{i\theta} = x + iy.$$

As a straight line has two perpendiculars in the plane of x, y, z (a plane being defined as the continuum perpendicular to a straight line), a straight angle is equal to three right angles.

The sum of the 3 $\angle$s of a triangle
= a straight angle
= 3 right angles.



The only results of a three-dimensional geometry which do not hold for four-dimensional geometry are the Pythagorean theorem and deductions from it.

Through a point P draw parallels to x, y, z axes, which are perpendicular to each other. Let u be the line perpendicular to x, y, z. A perpendicular makes either one right angle or two right angles. The parallels through P cut the x, y, z axes in six points, but yield only three sets each of two equal lengths.

There are two perpendiculars from P on OX.

They are equal as $\angle\ PM_2N_1 = 1$ right angle

$$= \angle\ PN_1M_2.$$

Also $M_2\,N_1 = PM_2 = PN_1$, as $PM_2\,N_1$ has all its angles equal, each being a right angle.

Also

$$\begin{aligned} z &= PL_2 \\ &= ON_1 \\ &= OM_2 + M_2\,N_1 \\ &= PL_1 + PM_2 \\ &= y + x. \end{aligned}$$

Let a hyperplane be the continuum perpendicular to two perpendicular lines.

Let r denote the distance OP in the hyperplane of OM_2P (perpendicular to z and u).

Let ρ denote the distance OP in the hyperplane of ON_1P (perpendicular to y and u).

Let θ be the angle between OX and r as stated.

Let ϕ be the angle between OX and

$$\begin{aligned} z &= x + y, \\ y &= z - x, \\ y &= OM_2,\ z = ON_1. \end{aligned}$$

As we took $r E^{\theta} = x + y = z = ON_1$,
so we can take

$$\begin{aligned} \rho\,.\,E^{\phi} &= OM_2 \\ &= y = z - x. \\ \therefore\ \rho\,.\,E^{\wedge\phi} &= z - (\wedge x), \\ \rho\,.\,E^{\forall\phi} &= z - (\forall x). \\ \therefore\ \rho^3 &= z^3 + (-x)^3 \\ &= z^3 - x^3. \end{aligned}$$

Hence there are two co-variants associated with two points O and P.

If P be a point, whose y and z are both zero,

$$r^3 = x^3, \quad \rho^3 = -x^3$$

i.e.,

$$r = x, \quad \rho = -x.$$

OP has then two co-variants x and $-x$, as on a line of three dimensional space.

If P be a point on OX, then $x = 0$, $r = \rho$.

But if P be outside the lines OX, OY, OZ, it has two co-variants r, ρ given by

$$r^3 = x^3 + y^3,$$
$$\rho^3 = z^3 - x^3.$$

These are the two co-variants associated with a plane of four dimensional space corresponding to the co-variant

$$r^2 = x^2 + y^2$$

of three-dimensional space.

We can take x to denote time, and y to denote space, then r denotes space.

We can take z to denote matter, then ρ denotes matter.

Einstein's relation between a three-dimensional space and time is a relation between matter and time and corresponds to

$$\rho^3 = z^3 - x^3.$$

Reduced to three-dimensional numbers of matter by the equation

$$k_1 r^2 = k_2 m_1^3,$$

the above relation yields

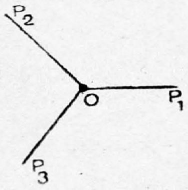
$$d\rho_1^2 = dz_1^2 - dx_1^2$$

which corresponds to $(ds^2 - dt^2)$.

Triations.

The three directions of a straight line are the positive, dinitive and trinitive.

$$(+1)^3 = (\wedge 1)^3 = (\mathsf{A}\, 1)^3 = 1.;$$
$$(\wedge 1)^2 = \mathsf{A}\, 1;$$
$$(\mathsf{A}\, 1)^2 = \wedge 1.$$



Measure three equal lengths OP_1, OP_2, OP_3 along the three directions of a straight line.

If $OP_1 = 1$, then $OP_2 = \wedge 1$, $OP_3 = \mathsf{A}\, 1$.

Each direction is the reverse of the other two : $\wedge$ 1 or ⩜ 1 being the reverse of 1.

OP_3 is the reverse of OP_1 and OP_2.

If OP_1 represents "to be" and OP_2 represents "not to be," then OP_3 is "both not to be and to be".

Hence in place of our three-dimensional logic which states that A is either B or not B, we get A is 'either B' or 'not B' or 'both B and not B.'

Again, the identity A = A holds in three-dimensional space. In four-dimensional space, we can take the fundamental identity as

$$A = A \parallel A.$$

i.e., A is equal to A relatively to A.

This is a synthesis of a congruence and what Dr. Whitehead calls a cogredience, *i.e.*, of the relation between two objects, and the relation between subject and object, these relations being given the same status.

The distinction between subject and object is transcended in this identity. I call it a *triation* (a three-sided equation).

We can manipulate changes in the sides of the identity thus :

If we change a quantity B from one side to another in direction, (*i.e.*, from right to left), we change sign by $\wedge$ (dinus) for one step and by ⩜ (trinus) for two steps. In the reverse direction it is by ⩜ for one step and by $\wedge$ for two steps.

Deductions in the same side of a triation are made by the negative sign. This is consistent with x and $-x$ being the two co-variants associated with O and a point, whose

$y = 0$, $z = 0$ in the plane $u = 0$.

Thus $$A = A \parallel A \qquad \ldots \quad \ldots \quad \ldots (1)$$

$$\therefore \quad A + B = (A + B) \parallel (A + B) \qquad \ldots \quad \ldots (2)$$

$$A \wedge B ⩜ B = (A - B) \parallel (A - B).$$

But $$\wedge B ⩜ B = - B.$$

$$\therefore \quad A - B = (A - B) \parallel (A - B) \qquad \ldots \quad \ldots (3)$$

$$A = A - B \parallel A ⩜ B$$

$$A \wedge B = A - B \parallel A$$

$$A = A + B \parallel A - (⩜ B)$$

$$= A + B \parallel A + B \wedge B.$$

From (1), (2) and (3) we see that an equation is a compressed triation with an invisible equal third side.

Our equations are two-sided and do not involve dinitive or trinitive quantities. We can always reduce a triation to a form, where each side involves only positive or negative quantities, but not dinitive or trinitive—which may be called the super-negative quantities. Then any two sides of the triation yield an equation.

The Law of Gravitation.

Newton's Law of Gravitation could be derived from the fundamental equation

$$k_1 r^2 = k_2 m^3.$$

The Einsteinian law is derived in terms of matter measures for time, space, matter and mind, from $t = k_1 r^2$.

The Law of Gravitation thus depends on the first three terms of my fundamental equations, that is given by

$$t = k_1 r^2 = k_2 m^3 ;$$

or writing it as a triation

$$t = k_1 r^2 \parallel k_2 m^3.$$

Reduce time and space to Euclidean continuum by taking

$$t_1^3 = t \text{ and } r_1^3 = r^2.$$

Hence we get

$$t_1 = k_1 r_1 \parallel k_2 m.$$

Here t_1, r_1, m are measured by rods of mind perpendicular to a continuum of time, space and matter.

Hence the Law of Gravitation can be written as

$$t = a_1 r \parallel a_2 m$$

where t, r, m are lengths of mind-stuff (*i.e.*, are four-dimensional numbers) and a_1, a_2 are constants.

This is a straight line in a three-directional plane given by

$$du = 0, \text{ or } u = \text{constant},$$

where u is mind.

Thus the Law of Gravitation, when expressed in terms of four-dimensional numbers is the analogue of Newton's First Law of Motion which is

$$t = kr.$$

The problem of n bodies is thus automatically solved in terms of four-dimensional numbers.

The Law of Gravitation means $\frac{dp}{dt}$ = constant, where dp is an element of the path of a particle.

This $\frac{dp}{dt}$ has three independent components $\frac{ds}{dt}$, $\frac{dm}{dt}$, $\frac{dL}{dt}$, each being constant.

$\frac{ds}{dt}$ = constant, gives the Law of Conservation of Velocity.

$\frac{dm}{dt}$ = constant, corresponds to the Law of Conservation of Matter and yields it, when the constant is zero.

$\frac{dL}{dt}$ = constant, corresponds to the Law of Conservation of Energy; for if L be a four-dimensional mass, $\frac{dL}{dt}$ is a three-dimensional mass as energy is.

Thus the Law of Gravitation involves as corollaries three laws which correspond to our laws of conservation of energy, matter and momentum.

Gravitation and Rotation.

We know that rotational motion is independent of translational motion in a three-dimensional space. Further it develops linearly. We can therefore regard it as a translation along a fourth-dimensional line perpendicular to the three-dimensional space.

But Gravitation also means a translation along a fourth dimensional line, as I have already shown.

Hence Gravitation and Rotation are both translatory motions along a fourth-dimensional line. This leads to Einstein's *Principle of Equivalence*.

Further, as a motion of translation is relative to the base of measurement, rotation is not absolute, but is relative (to a four-dimensional frame).

The general Principle of Relativity asserts that the laws of nature have a co-variant form with respect to four-dimensional axes. On my analysis, time, space, matter and mind furnish the four dimensions of a four-dimensional universe. Hence x, y, z, u, the four axes, represent time, *space*, *matter* and *mind*. A law of nature is a persistent relation between *mind* and an objective fact which is itself a relation of time, *space* and *matter*. Hence Einstein's principle reduces to this: a persistent relation between time *space*, *matter* and *mind* is a co-variant, *i.e.*, a persistent relation between x, y, z, u—which is a truism.

Miquel Theorems

BY V. RAMASWAMI AIYAR.

1. Let us consider a system of n elements $x_1, x_2, \ldots x_n$ in a plane which may be of any of the following types :—

(i) It may consist of n straight lines intersecting in pairs in distinct points of intersection.

(ii) It may consist of n circles intersecting in a point O and having their other intersections all distinct.

(iii) It may consist of $n-1$ circles and a straight line intersecting in a common point O and having their other intersections all distinct.

2. *Definition of Symbols and Statement of Theorems.*

By (x_1x_2) we shall mean, in the system of type (i), the point of intersection of the lines x_1, x_2; and, in systems of types (ii) and (iii), the point of intersection, other than O, of x_1, x_2.

By $(x_1x_2x_3)$ we shall mean the circle through the points (x_2x_3), (x_3x_1), (x_1x_2).

Then we have the following series of Miquel theorems :—

I. THEOREM FOR FOUR ELEMENTS. Let the system consist of 4 elements x_1, x_2, x_3, x_4. *Then the four circles denoted by* $(x_2x_3x_4)$, $(x_3x_4x_1)$, $(x_4x_1x_2)$ *and* $(x_1x_2x_3)$ *are all concurrent.*

The point of concurrence is denoted by $(x_1x_2x_3x_4)$.

II. THEOREM FOR FIVE ELEMENTS. Let the system consist of 5 elements x_1, x_2, x_3, x_4, x_5. *Then the five points denoted by* $(x_2x_3x_4x_5)$, $(x_3x_4x_5x_1)$, $(x_4x_5x_1x_2)$, $(x_5x_1x_2x_3)$ *and* $(x_1x_2x_3x_4)$ *all lie on a circle.*

This circle is denoted by $(x_1x_2x_3x_4x_5)$.

III. THEOREM FOR SIX ELEMENTS. Let the system consist of 6 elements $x_1, x_2, x_5, x_4, x_5, x_6$. *Then the six circles denoted by* $(x_2x_3x_4x_5x_6)$, $(x_3x_4x_5x_6x_1)$, ... $(x_1x_2x_3x_4x_5)$ *are all concurrent.*

This point of concurrence is denoted by $(x_1x_2x_3x_4x_5x_6)$.

IV. THEOREM FOR SEVEN ELEMENTS. Let the system consist of 7 elements x_1, x_2, ...x_7. *Then the seven points denoted by* $(x_2 x_3 \ldots x_7)$, $(x_3 x_4 \ldots x_1) \ldots (x_1 x_2 \ldots x_6)$ *all lie on a circle.*

This circle is denoted by $(x_1 x_2 \ldots x_7)$.

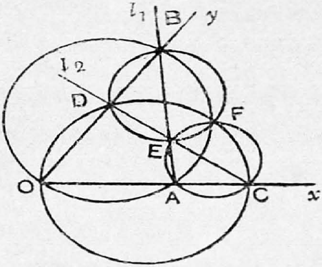
And the theorems go on in this way without limit.

I. Case of four elements.

3. Let us consider a system of four elements of type (i) consisting of the straight lines x, y, l_1, l_2. Then the theorem is that the circles (xyl_1), (xyl_2), (xl_1l_2) and (yl_1l_2) are concurrent. This is indeed well known and we shall not go into a proof of it. But, *from* its truth, we deduce a couple of *Rules of Operation*, as follows:—

Let O be the point of intersection of x, y and let us denote the circles (xyl_1), (xyl_2) by c_1, c_2. These have a point of intersection other than O. According to our notation it will be denoted by (c_1c_2). But this is the point of co-intersection of the four circles in question. Hence we have the statement of identity

$$(xyl_1l_2) = (c_1c_2) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$



which is one of our rules of operation.

Again we have

$$(xl_1l_2) = (xc_1c_2) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

for both symbols denote the circle ACFE in the figure; and this is the other rule of operation.

We observe that the truth of the theorem for four elements of types (ii) and (iii) follows from the truth for type (i) by inversion.

II. Case of five elements.

4. Let us next consider a system of five elements of type (i) consisting of the lines x, y, l_1, l_2, l_3. We have to prove that the points denoted by (xyl_2l_3), (xyl_3l_1), (xyl_1l_2), $(xl_1l_2l_3)$ and $(yl_1l_2l_3)$ lie on a circle.

As before let O be the point of intersection of x, y; and let the circles (xyl_1), (xyl_2), (xyl_3) be denoted by c_1, c_2, c_3.

Then, by the rule of operation (1), we have

$$\left.\begin{aligned}(xyl_2l_3) &= (c_2c_3)\\ (xyl_3l_1) &= (c_3c_1)\\ (xyl_1l_2) &= (c_1c_2).\end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

Let us now consider the point $(xl_1l_2l_3)$. It is the point of co-intersection of the circles $(l_1l_2l_3)$, (xl_2l_3), (xl_3l_1) and (xl_1l_2). Leaving the first of these circles alone, the point is sufficiently described as the point of co-intersection of the other three circles. But, by rule of operation (2), we have

$$\left.\begin{aligned}(xl_2l_3) &= (xc_2c_3)\\ (xl_3l_1) &= (xc_3c_1)\\ (xl_1l_2) &= (xc_1c_2).\end{aligned}\right\} \quad \ldots \quad \ldots \quad (4)$$

An inspection of the right-hand side shows that the point in question is that symbolized by $(xc_1c_2c_3)$. For x, c_1, c_2, c_3 form four elements of type (iii) and therefore the circles referred to in the right-hand side of (4), as well as the circle $(c_1c_2c_3)$, concur at the point $(xc_1c_2c_3)$. Hence we see that

$$\left.\begin{aligned}(xl_1l_2l_3) &= (xc_1c_2c_3);\\ \text{and similarly} \quad (yl_1l_2l_3) &= (yc_1c_2c_3);\end{aligned}\right\} \quad \ldots \quad \ldots \quad (5)$$

either of which now becomes a rule of operation. Review now the results (3) and (5). The left-hand sides in them consist of the five points under consideration and the right hand sides show that each of them lies on the circle $(c_1\ c_2\ c_3)$.

This proves the theorem for five elements of type (i) and gives us the rule of operation

$$(xy\ l_1\ l_2\ l_3) = (c_1\ c_2\ c_3). \quad \ldots \quad \ldots \quad (6)$$

The truth of the theorem for 5 elements of types (ii) and (iii) follows at once by inversion.

III. Case of six elements.

5. We shall first consider a system of six elements of type (i) consisting of the lines x, y, l_1, l_2, l_3, l_4. We have to prove that the circles denoted by $(xyl_2l_3l_4)$, $(xyl_3l_4l_1)$, $(xyl_4l_1l_2)$, $(xyl_1l_2l_3)$, $(xl_1l_2l_3l_4)$ and $(yl_1l_2l_3l_4)$ are all concurrent.

As before let O be the point of intersection of x, y and let the circle $(xylr)$ be denoted by c_r $(r = 1, 2, 3, 4)$. Then, by rule of operation (6) we have

$$\left.\begin{aligned}(xyl_2l_3l_4) &= (c_2c_3c_4)\\ (xyl_3l_4l_1) &= (c_3c_4c_1)\\ (xyl_4l_1l_2) &= (c_4c_1c_2)\\ (xyl_1l_2l_3) &= (c_1c_2c_3).\end{aligned}\right\} \quad \ldots \quad \ldots \quad (7)$$

The right-hand side makes it obvious that the four circles on the left-hand side are concurrent at the point denoted by $(c_1c_2c_3c_4)$. We infer this by the theorem for four elements applied to the system c_1, c_2, c_3, c_4 which is of type (ii).

From what has been proved above it really follows that *every four* out of the six circles in question are concurrent; for it merely depends on which two out of the six lines in question we select as x, y. It follows that all the six circles in question are concurrent.

But we may prove the concurrence of the remaining two circles, namely, $(xl_1l_2l_3l_4)$ and $(yl_1l_2l_3l_4)$, independently, as follows:—

Taking $(xl_1l_2l_3l_4)$, we see it is the circle on which lie the points $(l_1l_2l_3l_4)$, $(xl_2l_3l_4)$, $(xl_3l_4l_1)$, $(xl_4l_1l_2)$ and $(xl_1l_2l_3)$. Leaving the first point alone, the circle is sufficiently expressed as the circle passing through the other four points. But, by the rule of operation (5), we find that the remaining points are identically the same as those denoted by $(xc_2c_3c_4)$, $(xc_3c_4c_1)$, $(xc_4c_1c_2)$ and $(xc_1c_2c_3)$ respectively. By the theorem for five elements applied to x, c_1, c_2, c_3, c_4, which form a system of type (iii), we see that these four points, as well as the point $(c_1\ c_2\ c_3\ c_4)$, lie on the circle denoted by $(x\ c_1\ c_2\ c_3\ c_4)$. We have thus shown that

$$\left.\begin{aligned}&(xl_1l_2l_3l_4) = (xc_1c_2c_3c_4).\\ \text{Similarly}\quad &(yl_1l_2l_3l_4) = (yc_1c_2c_3c_4).\end{aligned}\right\} \quad \ldots \quad (8)$$

Either of these results will now count as a rule of operation.

Reviewing (7) and (8) together we see that the six circles in question are all concurrent at the point $(c_1c_2c_3c_4)$.

This proves the theorem ; and also gives us the rule of operation

$$(xy\,l_1l_2l_3l_4) = (c_1c_2c_3c_4). \qquad \dots \qquad \dots \quad (9)$$

The theorem being proved for elements of type (i) its truth for six elements of types (ii) and (iii) follows by inversion.

IV. Case of seven elements.

6. We shall next consider the theorem for seven elements of type (i). Let the seven lines be x,y,l_1,l_2,l_3,l_4,l_5. We have to prove that the seven points, namely, five points of the type $(xyl_2l_3l_4l_5)$ and the points $(xl_1l_2l_3l_4l_5)$, $(yl_1l_2l_3l_4l_5)$, all lie on a circle.

As before let O be the point of intersection of x, y and let the circle (xyl_r) be denoted by C_r $(r = 1, 2, 3, 4, 5)$. Then by the operation theorem (9), we have

$$\left.\begin{aligned} (xyl_2l_3l_4l_5) &= (c_2c_3c_4c_5), \\ (xyl_3l_4l_5l_1) &= (c_3c_4c_5c_1), \\ \dots\quad &\quad \dots \\ (xyl_1l_2l_3l_4) &= (c_1c_2c_3c_4). \end{aligned}\right\} \qquad \dots \qquad \dots \quad (10)$$

The right-hand side shows that these five points lie on the circle symbolized by $(c_1c_2c_3c_4c_5)$. We have really shown that *every five* out of the seven points in question lie on a circle; and it follows therefore that all the seven points in question are concyclic. But we can show independently that

$$\left.\begin{aligned} (xl_1l_2l_3l_4l_5) &= (xc_1c_2c_3c_4c_5), \\ \text{and} \qquad (yl_1l_2l_3l_4l_5) &= (yc_1c_2c_3c_4c_5), \end{aligned}\right\} \qquad \dots \qquad \dots \quad (11)$$

by using the rule of operation (8). [This result, in turn, becomes a rule of operation for the next stage].

From (10) and (11), we see in one view that all the seven points in question lie on the circle $(c_1c_2c_3c_4c_5)$, leading to the identity

$$(xyl_1l_2l_3l_4l_5) = (c_1c_2c_3c_4c_5), \quad \dots \qquad \dots \quad (12)$$

which becomes a rule of operation for the next stage.

The truth of the theorem for seven elements of types (ii) or (iii) now follows by inversion.

7. It will now be clear that we can carry on this reasoning to cover a system containing any number of elements; and thus the whole series of theorems stands proved.

8. Inverting with respect to an arbitrary point outside the plane, we see that the series of theorems hold for a system of circles on a sphere all having a common point O. The three types of plane elements considered in para. 1 lead but to this one type on the sphere.

9. Let us go a little further and consider a system of n circles in a plane co-intersecting in a point O, that is, a system of type (ii) of para 1. Let the circles be $x_1, x_2, \ldots x_n$. These will be sufficiently given if, in addition to the point O, we are granted the points diametrically opposite to O in the circles. Let the diametrically opposite points be $A_1, A_2, A_3, \ldots A_n$. It will at once be obvious that the point denoted by $(x_1\,x_2)$ for example is the foot of the perpendicular from O on A_1A_2. The circle $(x_1x_2x_3)$ thus becomes the pedal circle of O in respect of the triangle $A_1A_2A_3$. And we have a series of theorems as follows where, in each case, a fixed point O, as an origin of perpendiculars, is understood.

(1) Let there be four points A_1, A_2, A_3, A_4. Then the pedal circles of O with respect to the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$ and $A_1A_2A_3$ are concurrent. We symbolize the point of concurrence by $[A_1A_2A_3A_4]$.

(2) Let there be five points A_1, A_2, A_3, A_4, A_5. Then the five points arising, which are of the type $[A_2A_3A_4A_5]$, all lie on a circle. The circle may be denoted by $[A_1A_2A_3A_4A_5]$.

And the theorems go on this way, step by step, as additional points come in, without limit.

10. The reader will find the Miquel Theorems stated in Clifford's *Common Sense of the Exact Sciences.* The theorem for five elements of type (i) is the Miquel Theorem proper. The further development is perhaps due to Clifford himself, but the present writer is not sufficiently acquainted with the literature on the subject. However, he trusts, he has made the 'Common Sense' underlying the theorems sufficiently clear to the reader. It may be noted that it is all a matter lying within the range of the elementary geometry of the circle.

The Parametric Representation of the Twisted Cubic

By R. VAIDYANATHASWAMY.

The Twisted Cubic is a well-known curve and has been much studied both through parametric representation and by synthetic methods. The principal properties of the curve can be obtained straight-way by reducing it to the form

$$x_1 = t^3,\ x_2 = t^2,\ x_3 = t,\ x_4 = 1.$$

The object of this paper is however, to exhibit the symmetry of the results obtained, when we do not restrict ourselves to any special form of the parametric cubics. The method is based primarily on the properties of what I have called *apolar quartettes* of binary cubics. The parametric form of the equation of quadrics through the curve, and the interpretation ot the second transvectant obtained at the end of the paper are, I believe, new.

I have added a short introduction on the symbolic calculus, for the benefit of readers not familiar with the ideas of modern Invariant Theory.

It is necessary to observe that all the results of this paper are capable of extension to the Rational Normal Curve in N dimensions—though the intentionally special methods used here will not suffice in that case.

1. The Symbolic Method.

1. Let the binary n-ic

$$f(x) = a_0\, x_1{}^n + {}_nc_1\, a_1\, x_1{}^{n-1}\, x_2 + \ldots\ldots + a_n\, x_2{}^n$$

be written symbolically

$$\alpha_x{}^n = (\alpha_1\, x_1 + \alpha_2\, x_2)^n,$$

where α_1, α_2 are purely *fictitious* entities obeying the ordinary rules of algebra, but subject to the relations

$$\alpha_1{}^{n-r}\alpha_2{}^r = a_r.\quad (r = 0, 1\ldots n).$$

Any combination of α_1, α_2 which is not of the form $\alpha_1{}^{n-r}\ \alpha_2{}^r$ must be regarded as purely symbolic, *i. e.*, not possessing an actual significance. Thus the only products of the α's which correspond unambiguously to actual quantities are those of order n. As an example, the product

$$\alpha_1{}^n\alpha_2{}^n = \alpha_1{}^{n-r}\ \alpha_2 \,.\, \alpha_1{}^r\ \alpha_2{}^{n-r} = a_r\ a_{n-r}$$

for any r, and is therefore *ambiguous*. To represent unambiguously in

symbolic form functions of the a's of higher degree than the first, we must therefore have recourse to equivalent symbols α', α''......, where

$$f(x) = \alpha_x^n = (\alpha')_x^n = (\alpha'')_x^n = \ldots\ldots$$

The product $a_0\, a_n$, for instance, would then be represented by

$$\alpha_1^n \ (\alpha_2')^n = (\alpha_1')^n\, \alpha_2^n$$

(since α, α' are equivalent). Generally to represent a function of the rth degree in the a's, we have to use r equivalent symbols.

If we have more than one form, say

$$f(x) = \alpha_x^n = \alpha_x'^n = \ldots\ldots;\ g(x) = \beta_x^m = \beta_x'^m = \ldots \text{ etc.},$$

then any symbolic expression homogeneous in (α_1, α_2), (α_1', α_2') ... (β_1, β_2), (β_1', β_2')...has an actual significance, if it is of degree n in each pair of α's, of degree m in each pair of β's and so on.

It is shewn in the Theory of Invariants, that the most general covariant of two binary forms $f(x) = \alpha_x^n$, $g(x) = \beta_x^n$ which is linear in the co-efficients of each of them, must be of the shape $(\alpha\beta)^r\, \alpha_x^{n-r}\, \beta_x^{n-r}$ where, here and hereafter $(\alpha\beta)$ means $(\alpha_1\, \beta_2 - \alpha_2\, \beta_1)$. The expression $(\alpha\beta)^r\, \alpha_x^{n-r}\, \beta_x^{n-r}$ (which obviously has an actual significance) is called *the* r^{th} *transvectant* of the forms f and g and also written in the form $(f, g)^r$. It is easy to deduce the actual form of the rth transvectant from the symbolic form. For instance, $(\alpha\beta)^r\, \alpha_x^{n-r}\, \beta_x^{n-r}$ is a form of order $2n - 2r$, in which the co-efficient of x_1^{2n-2r} is

$$(\alpha\beta)^r\, \alpha_1^{n-r}\, \beta_1^{n-r} = \alpha_1^{n-r}\, \beta_1^{n-r} \left(\sum_k (-)^k\, {}_rC_k\, \alpha_1^{r-k} \alpha_2^k\, \beta_1^k \beta_2^{r-k}\right)$$

$$= \sum_k (-1)^k\, {}_rC_k\, a_k\, b_{r-k}.$$

if $$f(x) = \sum_k {}_nC_k\, a_k\, x_1^{n-k}\, x_2^k,\ \phi(x) = \sum_k {}_nC_k\, b_k\, x_1^{n-k}\, x_2^k.$$

If the form g is supposed identical with f, we have the *self-transvectants* of f. In this case, however, a peculiarity arises. If

$$f(x) = \alpha_x^n = \alpha_x'^n,$$

then the rth transvectant

$$(f, f)^r = (\alpha\alpha')^r\, \alpha_x^{n-r}\, \alpha_x'^{n-r}$$

and since α, α' refer to the same form, they are equivalent and can therefore be interchanged without altering the value of any expression. Hence

$$(f, f)^r = (\alpha\alpha')^r \alpha_x^{n-r} \alpha_x'^{n-r} \equiv (\alpha'\alpha)^r \alpha_x^{n-r} \alpha_x'^{n-r}$$

But $$(\alpha'\alpha) = (\alpha_1'\alpha_2 - \alpha_2'\alpha_1) = -(\alpha\alpha').$$

Hence

$$(\alpha\alpha')^r \alpha_x^{n-r} \alpha'^{n-r}_x = (-1)^r (\alpha'\alpha)^r \alpha_x^{n-r} \alpha'^{n-r}_x.$$

Hence $(f, f)^r$ must be zero identically, if r is odd. In other words *the transvectants of odd index of any form will vanish identically.*

2. The most important transvectants are the first, the second and the last. The first transvectant

$$(f, g)^1 = (\alpha\beta)\, \alpha_x^{n-1} \beta_x^{n-1}$$

differs only by a numerical multiple from the Jacobian

$$\frac{df}{dx_1}\frac{dg}{dx_2} - \frac{df}{dx_2}\frac{dg}{dx_1}$$

of f and g. In fact by ordinary differentiation, we have

$$\frac{d\alpha_x^n}{dx_1}\frac{d\beta_x^n}{dx_2} - \frac{d\alpha_x^n}{dx_2}\frac{d\beta_x^n}{dx_1} = n(\alpha_1\beta_2 - \alpha_2\beta_1)\, \alpha_x^{n-1} \beta_x^{n-1}$$
$$= n\,(f, g)^1.$$

On this account the first transvectant of two forms is *called* their Jacobian.

The second transvectant $(f, f)^2$ of f with itself is called the Hessian of f and may be easily shewn to be a numerical multiple of the determinant

$$\begin{vmatrix} \dfrac{d^2f}{dx_1^2}, & \dfrac{d^2f}{dx_1dx_2} \\ \dfrac{d^2f}{dx_1dx_2}, & \dfrac{d^2f}{dx_2^2} \end{vmatrix}$$

An important property of the Hessian is given by the theorem :

The identical vanishing of the Hessian of **f** *is the necessary and sufficient condition that* **f** *may be the* n^{th} *power of a linear form.**

* For proof : see Grace and Young : *Algebra of Invariants*, page 235.

Lastly the nth transvectant of f and g is their apolar or harmonic invariant

$$a_0 b_n - {}_nC_1 a_1 b_{n-1} \ldots + (-1)^n a_n b_0.^*$$

If this invariant vanishes, the forms f, g are said to be *apolar.* Since $(f, f)^r$ vanishes if r is odd, it follows that :—*A form of odd order is necessarily apolar to itself.* †

Another important theorem which we may easily prove, is: *The only linear forms whose n^{th} powers are apolar to* **f** *are the linear factors of* **f** *itself.*

For when a_x^n is the nth power of a linear form a_x, we may regar the a in a_x^n indifferently as a symbol or as an actual quantity. The condition of apolarity of a_x^n and $f = \alpha_x^n$ is then

$$(\alpha a)^n = (\alpha_1 a_2 - \alpha_2 a_1)^n = 0;$$

which shews that $f(x)$ vanishes when $x_1 = a_2$, $x_2 = -a_1$ and therefore contains the factor a_x.

II. Apolar sets of Binary Cubics.

3. Let

$$\begin{aligned} f_1(x) &= \alpha x^3 = a_0 x_1^3 + 3a_1 x_1^2 x_2 + \quad \ldots \quad + a_3 x_2^3. \\ f_2(x) &= \beta x^3 = b_0 x_1^3 + 3b_1 x_1^2 x_2 + \quad \ldots \quad + c_3 x_2^3. \\ f_3(x) &= \gamma x^3 = c_0 x_1^3 + 3c_1 x_1^2 x_2 + \quad \ldots \quad + c_3 x_2^3. \\ f_4(x) &= \delta x^3 = d_0 x_1^3 + 3d_1 x_1^2 x_2 + \quad \ldots \quad + d_3 x_2^3. \end{aligned}$$

be four linearly independent binary cubics. Let us denote the Jacobian $(f_r, f_s)^1$ by J_{rs} and the harmonic invariant $(f_r, f_s)^3$ by i_{rs}; so that

$$J_{rs} = -J_{sr} \text{ and } i_{rs} = -i_{sr}.$$

Since any 6 binary quartics must be connected by a linear relation, it follows that there is an identical linear relation between J_{23}, J_{31}, J_{12}, J_{14}, J_{24}, J_{34}. We can shew that this relation is

$$i_{14} J_{23} + i_{24} J_{31} + i_{34} J_{12} + i_{23} J_{14} + i_{31} J_{24} + i_{12} J_{34} \equiv 0. \qquad \text{(A)}$$

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* Compare : 'Linear Systems of Points', J.I.M.S., Vol. XI, p. 100.

† Compare : *Ibid.*

For, the symbolic expression of the left-hand side is

$$(\alpha\delta)^3(\beta\gamma)\beta_x^2\gamma_x^2 + (\beta\delta)^3(\gamma\alpha)\gamma_x^2\alpha_x^2 + (\gamma\delta)^3(\alpha\beta)\alpha_x^2\beta_x^2$$
$$+ (\beta\gamma)^3(\alpha\delta)\alpha_x^2\delta_x^2 + (\gamma\alpha)^3(\beta\delta)\beta_x^2\delta_x^2$$
$$+ (\alpha\beta)^3(\gamma\delta)\gamma_x^2\delta_x^2.$$

Regarding the α's, β's etc. momentarily as actual algebraic quantities, this expression is a cubic in α, and must therefore vanish identically, if it is shewn to vanish for *four* values of α. Now when $\alpha=\delta$ (which means of course $\alpha_1=\delta_1$, $\alpha_2=\delta_2$) the expression becomes

$$(\beta\delta)(\gamma\delta)\delta_x^2\{(\beta\delta)^2\gamma_x^2-(\beta\delta)^2\gamma_x^2+(\gamma\delta)^2\beta_x^2$$
$$-(\beta\delta)^2\gamma^2_x\} = 0.$$

Similarly, it vanishes when $\alpha = \gamma$ or β. Finally, if $\alpha_1 = x_2$, $\alpha_2 = -x_1$, it becomes

$$\beta_x^2\gamma_x^2\delta_x^2\ \{(\beta\gamma)\delta_x-(\beta\delta)\gamma_x+(\gamma\delta)\beta_x\} = 0,$$

since the expression within the brackets is identically zero.

Thus the expression vanishes identically when α, β, γ, δ, x represent arbitrary algebraic quantities. Hence *the expression will also vanish identically when some or all of* α, β, γ, δ, x *represent entities other than algebraic quantities, provided they continue to obey all the formal laws of algebra**, and in particular, when α, β, γ, δ are symbols of the invariant calculus. Thus the truth of the identity (A) is established.

4. Since the apolar invariant of two cubics is linear in the co-efficients of each of them, it follows that a cubic can be uniquely determined (apart from an arbitrary numerical multiplier) so as to be apolar to three given linearly independent cubics. Let F_1, F_2, F_3, F_4 be the cubics thus determined, so as to be apolar to the cubics f_1, f_2, f_3, f_4 three and three (*i.e.* F_1 is apolar to f_2, f_3, f_4 and so on). To fix the arbitrary numerical multipliers in F_1, F_2, F_3, F_4, let us suppose them so determined that

$$(f_1, F_1)^3 = (f_2, F_2)^3 = (f_3, F_3)^3 = (f_4, F_4)^3.$$

* The essence of algebra is its formalism, by virtue of which it is able to disregard the special concrete character which the entities behind its symbols may possess. In fact, these entities may be anything whatever under the sun, provided only they are susceptible of certain kinds of relationships typified by the operations of algebra. To narrow them down to mere numbers, real or complex, would be indeed not to recognise the deep inwardness of algebraic symbolism.

It is this fact of formalism which is expressed, in an exaggerated form in Russell's *famous definition* of mathematics

The two quartettes of binary cubics (F_1, F_2, F_3, F_4), (f_1, f_2, f_3, f_4) which satisfy the relations

$$(F_r, f_s)^3 = 0, \; (F_r, f_r)^3 = (F_s, f_s)^3 \quad (r \neq s)$$

will be termed the *apolar quartettes* of each other.

The fundamental property of such apolar quartettes and the source of all the results of this paper is the identity

$$f_1(x)F_1(y) + f_2(x)F_2(y) + f_3(x)F_3(y) + f_4(x)F_4(y) \equiv k'(xy)^3,$$

where k' is an invariant of the f's and $(xy) = (x_1y_2 - x_2y_1)$.

We shall assume the truth of this identity here without attempting an algebraical proof of it; as we shall see later on, it is capable of a very simple geometrical proof. From this fundamental identity we shall now deduce three others, which we shall refer to as identities (B), (C), and (D).

If we regard $\Sigma_r f_r(x) F_r(y)$ as a form in y, then since it is a perfect cube, its Hessian ought to vanish identically; that is

$$\Sigma f_r(x)^2 \{F_r(y), F_r(y)\}^2 + 2\Sigma f_r(x) f_s(x) \{F_r(y), F_s(y)\}^2 \equiv 0 \quad \text{(B)}$$

This identity gives a general quadratic relation between the four cubics $f_r(x)$. Since this is an identity also in y, it will remain true when y_1^2, y_1y_2, y_2^2 are replaced by arbitrary parameters λ, μ, ν. When we do this we obtain an identity which we shall call (C), which gives a quadratic relation between the cubics $f_r(x)$, more general than the identity (B). Indeed, it can be proved from geometrical considerations that (C) gives the most general such relation.

If β_y^3 be a perfect cube, so that β is both a symbol and an actual quantity, then its second transvectant with any other cubic $\phi(y) = \alpha^3_y$ is $(\beta\alpha)^2 \beta_y \alpha_y$ and therefore contains the factor β_y (which has an actual significance since β is an actual quantity in addition to being symbolic). Hence, since $\Sigma f_r(x) F_r(y)$, regarded as before as a form in y, is a perfect cube $(xy)^3$, its second tranvectant with an arbitrary cubic $\phi(y)$ must contain the factor (xy). Thus $\Sigma f_r(x) \{F_r(y), \phi(y)\}^2$ contains the factor (xy) and therefore vanishes when $x = y$; so that we have

$$\Sigma f_r(y) \{F_r(y), \phi(y)\}^2 \equiv 0 \quad \ldots \quad \ldots \quad \text{(D)}.$$

5. The members F_1, F_2, F_3, F_4 of the apolar set admit of simple symbolic representation in terms of the f's. Thus if $f_1(x) = \alpha_x^3$ etc. then

$$F_1(x) = \pm(\beta\gamma)(\gamma\delta)(\delta\beta)\beta_x \gamma_x \delta_x.$$

But as it is not proposed to use the symbolic calculus in detail here, we prefer to express the F's linearly in terms of the f's, (which is certainly possible since the f's are linearly independent) To obtain the expressions, suppose

$$F_p = \Sigma_q \lambda_{pq} f_q.$$

Then since $(f_r, F_r)^3 = 0$ $(r \mp p)$ and $(f_r, F_r)^3 = (f_p, F_p)^3$, it follows that λ_{pq} is proportional to I_{pq}, the minor of i_{pq} in the determinant

$$I = | i_{pq} | .$$

The determinant I is equal to $(i_{23} i_{14} + i_{31} i_{24} + i_{12} i_{34})^2$ and $I_{23}, I_{31}, I_{12}, I_{14}, I_{24}, I_{34}$ are easily found to be proportional to $i_{14}, i_{24}, i_{34}, i_{23}, i_{31}, i_{12}$ respectively.

We have therefore

$$F_1 = I_{12} f_2 + I_{13} f_3 + I_{14} f_4 = k (i_{34} f_2 - i_{24} f_3 + i_{23} f_4) \text{ etc.,}$$

where k is a constant.

6. We may notice one further property of apolar sets, viz., *Any combinant* of the forms f differs only by a constant multiplier from a corresponding combinant of the F's.*

For instance, since

$$F_1 = i_{34} f_2 - i_{24} f_3 + i_{23} f_4,$$

$$F_2 = i_{34} f_1 + i_{14} f_3 + i_{31} f_4,$$

$$(F_1, F_2)^3 = i_{34} (i_{23} i_{14} + i_{31} i_{24} + i_{12} i_{34}) = k' (f_3, f_4)^3.$$

And the Jacobian

$$\begin{aligned}(F_1, F_2)^1 &= i_{34} (i_{23} J_{14} + i_{31} J_{24} + i_{14} J_{23} \\ &\qquad + i_{24} J_{31} + i_{34} J_{12}) - i_{24} i_{31} J_{34} - i_{23} i_{14} J_{34} \\ &= -(i_{23} i_{14} + i_{31} i_{24} + i_{12} i_{34}) J_{34}, \qquad \text{[From (A)]} \\ &= k'' (f_3, f_4)^1.\end{aligned}$$

* The combinants of r binary n-ics, $f_1 f_2 \ldots f_r$ are those covariants, which, when $f_1 f_2 \ldots$ are replaced by arbitrary linear combinations of themselves reproduce themselves multiplied by a power of the determinant of the substitution.

The transvectants of *odd* index of two binary forms are easily seen to be combinants of the forms.

III. Parametric Representation of the Twisted Cubic.

7. After the study of the properties of apolar sets of binary cubics, the development of the geometry of the twisted Cubic from its parametric representation offers no difficulty. Thus, let us take the parametric equations of the curve in the form :

$$x_1 = f_1(t),\ x_2 = f_2(t),\ x_3 = f_3(t),\ x_4 = f_4(t).$$

The co-ordinates l_{rs} of the tangent at any point t of the curve, are obviously the determinants of the matrix

$$\left\| \begin{matrix} \frac{df_1}{dt_1} & \frac{df_2}{dt_1} & \frac{df_3}{dt_1} & \frac{df_4}{dt_1} \\ \frac{df_1}{dt_2} & \frac{df_2}{dt_2} & \frac{df_3}{dt_2} & \frac{df_4}{dt_2} \end{matrix} \right\|$$

that is, they are the Jacobians J_{rs} of the cubics $f_1\ f_2\ f_3\ f_4$, two and two. Hence, by virtue of identity (A), we see that the tangents of the curve belong to the linear complex

$$i_{14}l_{23} + i_{24}l_{31} + i_{34}l_{12} + i_{23}l_{14} + i_{31}l_{24} + i_{12}l_{34} = 0 \qquad \ldots \quad (A')$$

If I_{rs} is the minor of i_{rs} in $|\ i_{rs}\ |$, this equation might be written in the more symmetrical form

$$\Sigma(I_{rs}\, l_{rs}) = 0. \qquad \ldots \qquad \ldots \quad (A')$$

When we replace the line-co-ordinates in the equation by their expressions in terms of point-co-ordinates, we obtain the bilinear relation

$$\Sigma I_{rs}\,(x_r y_s - x_s y_r) = 0.$$

This relation determines a polarity (A′) which replaces each point (x_1, x_2, x_3, x_4) by its polaror *nul*-plane with respect to the complex (A′), namely the plane whose co-ordinate are

$$L_r = I_{r1}x_1 + I_{r2}x_2 + I_{r3}x_3 + I_{r4}x_4 \quad (r = 1, 2, 3, 4),$$

and reciprocally each plane (l_1, l_2, l_3, l_4) by its *nul*-point with respect to the complex (A′), namely the point

$$x_r = i_{r1}l_1 + i_{r2}l_2 + i_{r3}l_3 + i_{r4}l_4. \qquad (r = 1, 2, 3, 4).$$

8. The co-ordinates L_1, L_2, L_3, L_4 of the osculating plane at the point t of the curve, are obviously determinants of the matrix,

$$\begin{vmatrix} \dfrac{d^2 f_1}{dt_1^2} & \dfrac{d^2 f_2}{dt_1^2} & \dfrac{d^2 f_3}{dt_1^2} & \dfrac{d^2 f_4}{dt_1^2} \\ \dfrac{d^2 f_1}{dt_1 dt_2} & \dfrac{d^2 f_2}{dt_1 dt_2} & \dfrac{d^2 f_3}{dt_1 dt_2} & \dfrac{d^2 f_4}{dt_1 dt_2} \\ \dfrac{d^2 f_1}{dt_2^2} & \dfrac{d^2 f_2}{dt_2^2} & \dfrac{d^2 f_3}{dt_2^2} & \dfrac{d^2 f_4}{dt_2^2} \end{vmatrix}$$

It is easily verified that these determinants are the cubics F_1, F_2, F_3, F_4 which form the apolar set of f_1, f_2, f_3, f_4. Thus the plane equations of the curve are

$$L_r = F_r(t). \quad [r = 1, 2, 3. 4].$$

The osculating planes which go through a given point (x_1, x_2, x_3, x_4) are therefore determined by the equation

$$\Sigma x_r F_r(t) = 0,$$

which shews that three and only three osculating planes pass through any point, or *the curve is of class* 3.

Since the curve is also of order 3, it follows that through the point t of the curve, there goes no other osculating plane than the osculating plane at t. Hence, the expression

$$f_1(t) F_1(t') + f_2(t) F_2(t') + f_3(t) F_3(t') + f_4(t) F_4(t')$$

vanishes if and only if $t = t'$: that is we must have

$$\Sigma f_r(t) F_r(t') \equiv k (tt')^3,$$

which is the fundamental property of apolar sets assumed before.

The polar plane of any point (x_1, x_2, x_3, x_4) with respect to (A′), the linear complex of tangents of the curve, was shewn to be the plane

$$L_r = \sum_s I_{rs} x_s.$$

Since we have also the linear relations

$$F_r = \sum_s I_{rs} f_s$$

it follows that the polar plane of any point of the curve with respect to (A′) is the osculating plane thereat. Since, in view of the symmetry of the equation of (A′), the polar plane of x passes through y whenever the polar plane of y passes through x, it follows that the polar plane of any point x with respect to (A′) is the plane determined by the points of contact of the three osculating planes which pass through x.

Notes and Questions.

On a Theorem in Higher Plane Curves.

The object of this note is to show that a certain theorem in Higher Plane Curves (See § 83, *Higher Plane Curves*, by Mr. S. M. Ganguli, M.Sc., and Hilton: *Plane Algebraic Curves*, p. 91, Ex. 11) requires some modification and is capable of further generalisation. The theorem is as follows:—On the polar line of any point A there are three points the first polars of which have a point of inflection at that point.

The proof of the theorem shows where modification is necessary.

Let B (x_2, y_2, z_2) be a point on the polar line of A (x_1, y_1, z_1), viz.

$$x\frac{\partial f}{\partial x_1} + y\frac{\partial f}{\partial y_1} + z\frac{\partial f}{\partial z_1} = 0 \qquad \ldots \qquad \ldots (1)$$

The first polar of B, viz.

$$F \equiv x_2\frac{\partial f}{\partial x} + y_2\frac{\partial f}{\partial y} + z_2\frac{\partial f}{\partial z} = 0 \qquad \ldots (2)$$

will have an inflection at A, if the Hessian of F

$$\begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix} = 0 \qquad \ldots \qquad \ldots (3)$$

passes through A, in which case the intersections of (3) regarded as a cubic in x_2, y_2, z_2 with (1) will give three points the first polars of which have an inflection at A.

We cannot see how the proof gets modified if we replace the point of inflection by a double point. Even if A is a double point on $F = 0$, the Hessian of F passes through A.

It is therefore necessary to exclude such cases and state explicitly that A does not lie on the Hessian of the given curve. In that case, it is certain that the first polar of none of the three points determined can have a double point at A and the above investigation holds.

If however A is a point on the Hessian of the given curve, the polar line of A given by (1) touches the Steinerian at a point, say, B (x_2, y_2, z_2)

and A and B are conjugate points. We shall show that in that case the cubic given by (3) passes through B.

Multiplying the columns of (3) by x_2, y_2, z_2 respectively, and adding, the first column of the resulting determinant vanishes, since A and B are conjugate points. Hence (3) regarded as a cubic passes through B and intersects (1) in two other points, the first polars of which have each an inflection, while that of B has a double point at A.

The theorem may therefore be stated as follows :—

Theorem I. There are three or two points in the plane, the first polars of which with respect to a given n-ic, have a point of inflection at an assigned point, according as the assigned point does not or does lie on the Hessian of the n-ic. Moreover, these points lie on the polar line of the assigned point.

A given line has $[(n-1)^2 - \delta - 2k]$ proper poles with respect to a given n-ic with δ nodes and k cusps and any one of these poles may be the assigned point in the above theorem.

We therefore deduce

Theorem II. There are $(n-1)^2 - \delta - 2k$ sets of three points each on any line not touching the Steinerian such that the first polars of the members of any set with respect to the n-ic with δ nodes and k cusps have a point of inflection at some one of the proper poles of the given line.

In the above, since the given line does not touch the Steinerian, none of its proper poles can lie on the Hessian and hence each set consists of three points the first polar of each of which has a point of inflection at some one of the proper poles of the line.

But if the given line L touches the Steinerian at a point P, the first polars of any two points on L touch at a point Q on the Hessian and intersect in $[(n-1)^2 - 2]$ points none of which can lie on the Hessian. For, if possible, let one of these points say Q_1 lie on the Hessian. Then, the given line L (being the polar line of Q, as well) must touch the Steinerian at another point P_1, or else P_1 and P coincide. But we have taken the line as touching the Steinerian at only one point and hence the first alternative does not hold. The second alternative requires that one and the same point on the Steinerian should be the conjugate point of two distinct ordinary points on the Hessian and L has a four-point contact with the Steinerian. Hence none of the $[(n-1)^2 - 2]$ poles can lie

on the Hessian. Moreover, we deduce incidentally that if L touches the Steinerian at p points, p of the poles lie on the Hessian and none of the remaining $(n-1)^2-2p$ poles can lie on the Hessian.

We therefore deduce

Theorem III. Given any line touching the Steinerian at p points, there are

(i) $(n-1)^2-\delta-2k-2p$ sets of three points each such that the first polars of the members of any set with respect to a given n - ic with δ nodes and k cusps, have a point of inflection at some one of the $(n-1)^2-\delta-2k-2p$ proper poles not lying on the Hessian; and

(ii) p other sets of three points each such that of the first polars of the members of any set one has a double point and each of the other two a point of inflection at some one of the p proper poles lying on the Hessian.

M. LAKSHMANAMURTI.

A certain Locus in Tetrahedral Co-ordinates.

If the tetrahedral co-ordinates of P be $(\alpha_1, \beta_1, \gamma_1, \delta_1,)$ the equations of the straight line whose direction angles are $\theta_1, \theta_2, \theta_3, \theta_4$ are

$$\frac{\alpha-\alpha_1}{l}=\frac{\beta-\beta_1}{m}=\frac{\gamma-\gamma_1}{n}=\frac{\delta=\delta_1}{p}=\rho$$

where $\qquad l=a\cos\theta_1,\ m=b\cos\theta_2,\ n=c\cos\theta_3,\ p=d\cos\theta_4$

and $\qquad l+m+n+p=0.$

It is easy to see that the co-ordinates of the feet of the perpendiculars on the faces $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ of the tetrahedron of reference are

$$0,\quad \beta_1-\frac{m_1}{l_1}\alpha_1,\quad \gamma_1-\frac{n_1}{l_1}\alpha_1,\quad \delta_1-\frac{p_1}{l_1}\alpha_1;$$

$$\alpha_1-\frac{l_2}{m_2}\beta_1,\quad 0,\quad \gamma_1-\frac{n_2}{m_2}\beta_1,\quad \delta_1-\frac{p_2}{m_2}\beta_1;$$

$$\alpha_1-\frac{l_3}{n_3}\gamma_1,\quad \beta_1-\frac{m_3}{n_3}\gamma_1,\quad 0,\quad \delta_1-\frac{p_3}{n_3}\gamma_1;$$

$$\alpha_1-\frac{l_4}{p_4}\delta_1,\quad \beta_1-\frac{m_4}{p_4}\delta_1.\quad \gamma_1-\frac{n_4}{p_4}\delta_1,\quad 0.$$

The feet will lie on a plane, if

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ l_1\alpha_1 - l_1\alpha_1 & l_1\beta_1 - m_1\alpha_1 & l_1\gamma_1 - n_1\alpha_1 & l_1\delta_1 - p_1\alpha_1 \\ m_2\alpha_1 - l_2\beta_1 & m_2\beta_1 - m_2\beta_1 & m_2\gamma_1 - n_2\beta_1 & m_2\delta_1 - p_2\beta_1 \\ n_3\alpha_1 - l_3\gamma_1 & n_3\beta_1 - m_3\gamma_1 & n_3\gamma_1 - n_3\gamma_1 & n_3\delta_1 - p_3\gamma_1 \\ p_4\alpha_1 - l_4\delta_1 & p_4\beta_1 - m_4\delta_1 & p_4\gamma_1 - n_4\delta_1 & p_4\delta_1 - p_4\delta_1 \end{vmatrix} = 0.$$

This determinant breaks up into sixteen determinants twelve of which vanish identically and the other four may be written

$$(1468) + (2368) + (2458) + (2467) = 0.$$

Remembering that

$$l + m + n + p = 0;\ l_1 = a,\ m_2 = b,\ n_3 = c,\ p_4 = d;$$

$$\frac{m_1}{b} = \frac{l_2}{a},\ \frac{n_1}{c} = \frac{l_3}{a};\ \frac{p_1}{d} = \frac{l_4}{a},\ \frac{n_2}{c} = \frac{m_3}{b},\ \frac{p_2}{d} = \frac{m_4}{b},\ \frac{p_3}{d} = \frac{n_4}{c};$$

which lead to

$$\begin{aligned} al_1 + bl_2 + cl_3 + dl_4 &= 0, \\ am_1 + bm_2 + cm_3\ dm_4 &= 0, \\ an_1 + bn_2 + cn_3 + dn_4 &= 0, \\ ap_1 + bp_2 + cp_3 + dp_4 &= 0, \end{aligned} \qquad (1)$$

we see that the points lie in a plane, if

$$\alpha_1\beta_1\gamma_1\delta_1\left(\frac{a^2}{\alpha_1} + \frac{b^2}{\beta_1} + \frac{c^2}{\gamma_1} + \frac{d^2}{\delta_1}\right) \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ l_4 & m_4 & n_4 & p_4 \end{vmatrix} = 0.$$

But from (1), the determinant is not zero, unless

$$\frac{\alpha_1}{l_1} = \frac{\beta_1}{m_1} = \frac{\gamma_1}{n_1} = \frac{\delta_1}{p_1} = \frac{\alpha_1 + \beta_1 + \gamma_1 + \delta_1}{l_1 + m_1 + n_1 + p_1};$$

that is, unless P lies on the plane at infinity $\alpha + \beta + \gamma + \delta = 0$.

Hence we have in quadriplanar co-ordinates:

Theorem. The locus of a point, the feet of the perpendiculars from which on the faces of the tetrahedron of reference lie in a plane is the cubic surface

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} + \frac{d}{\delta} = 0.$$

This surface is easily seen to be the locus of the focus of a paraboloid of revolution touching the faces of the tetrahedron of reference.

Hence the two loci are identical.

HEMRAJ.

Note on some Criteria for Divisibility. [J. I. M. S., Vol. XIV, No. 2, p. 68.]

The criteria given by Mr. Das amount to nothing more than actual division backwards from the unit's place of the dividend. For example, to test the divisibility of 1937 by 149, we may proceed with the division as indicated below :

$$\begin{array}{r|l} 13 & \\ 1937 & 149 \\ 447 & \\ \hline 149 & \\ 149 & \\ \hline \end{array}$$

The method consists in finding a suitable *partial* quotient at each step which, when multiplied by the divisor, gives the same unit's digit as in the *partial* remainder. Mr. Das's method is a slight modification of the above.

A. A. KRISHNASWAMI AYYANGAR.

Remarks on the Perpetual Calendar designed by Mr. C. Ganapati Aiyar (p. 70., J. I. M. S., Vol. XIV, No. 2).

The number of steps to the right or left that is to be counted in Mr. Ganapati's Tabular design may be easily calculated as the least positive residue (mod. 7) of

$$5(n-1)+\left[\frac{n}{4}\right]+p+\left[\frac{p}{4}\right]$$

where n stands for the century number, p for the year number and $\left[\frac{n}{4}\right]$ and $\left[\frac{p}{4}\right]$ denote the corresponding integral quotients.

Thus for the year 1815, the number of steps required is the least positive residue (mod. 7) of $(5\times17+4+15+3)$, *viz.*, 2.

If the least negative residue be taken, the steps should be counted to the left.

A. A. KRISHNASWAMI AYYANGAR.

Note on Question 1162.

(K. J. SANJANA) :—Solve the following differential equations and explain their geometrical significance :

(i) $x^2 + y^2 - \dfrac{2xy_1(1+y_1^2)}{y_2} + \dfrac{2y(1+y_1^2)}{y_2} = k^2$;

(ii) $x^3 + y^2 - \dfrac{2xy_1(1+y_1^2)}{y_2} + \dfrac{2y(1+y_1^2)}{y_2} + \dfrac{(1+y_1^2)^3}{y_2^2} = k^2$;

(iii) $y - mx + \dfrac{1+y_1^2}{y_2} + \dfrac{my_1(1+y_1^2)}{y_2} = 0.$

The solution of this question by Mr. C. N. Srinivas Iyengar, published in the issue of the Journal for June 1922, is defective. The analytical solution of the equations is not attempted and the geometrical interpretations are not quite general.

(i) The equation may be written :

$$(x^2 + y^2 - k^2)y_2 = 2(xy_1 - y)(1 + y_1^2). \qquad \ldots (1)$$

Differentiating and re-arranging, we have

$$(x^2 + y^2 - k^2)y_3 = 6y_1y_2(xy_1 - y). \qquad \ldots \qquad \ldots (2)$$

Dividing (2) by (1), $\dfrac{y_3}{y_2} = \dfrac{3y_1y_2}{1+y_1^2}$.

Integrating, we have $\log(cy_2) = \frac{3}{2}\log(1 + y_1^2)$

or $$cy_2 = (1+y_1^2)^{\frac{3}{2}}, \text{ i.e., } \frac{cy_2}{(1+y_1^2)^{\frac{3}{2}}} = 1.$$

Multiplying by y_1dx and again integrating, we have

$$\frac{c}{(1+y_1^2)^{\frac{1}{2}}} = y + c_1 :$$

$$\therefore \qquad y_1 = \frac{\pm\{(c^2 - (y+c_1)^2\}^{\frac{1}{2}}}{y+c_1},$$

whence we have $$dx = \pm\frac{(y+c_1)dy}{\{c^2 - (y+c_1)^2\}^{\frac{1}{2}}}.$$

On integration $$x + c_2 = \pm\{c^2 - (y+c_1)^2\}^{\frac{1}{2}}.$$

Squaring and arranging we find that the primitive is of the form

$$x^2 + y^2 + 2Ax + 2By + C = 0. \qquad \ldots \qquad \ldots (3)$$

We observe (1) that the equation obtained above represents loci which consist of circles only, and (2) that the constants A, B, C could not be all independent, as the given equation is of the second order.

Making the substitutions

$$\bar{x} = x - y_1 \frac{1 + y_1^2}{y_2}, \quad \bar{y} = y + \frac{1 + y_1^2}{y_2} \text{ and } \rho^2 = \frac{(1 + y_1^2)^3}{y_2^2},$$

we find that the differential equation reduces to

$$\bar{x}^2 + \bar{y}^2 - \rho^2 = k^2.$$

This shews that the curves represented must be such that the power of the origin with regard to the circle of curvature at any point is constant and equal to k^2, or (which is the same thing) that the circle of curvature at every point cuts orthogonally the circle $x^2 + y^2 = k^2$. As the primitive represents circles and as the circle of curvature at any point of a circle is the circle itself, we see that $C = k^2$ and a system of circles orthogonal to $x^2 + y^2 = k^2$ is represented.

Hence the primitive is

$$x^2 + y^2 + 2Ax + 2By + k^2 = 0.$$

When the centre of any such circle goes to infinity, that circle becomes a diameter of $x^2 + y^2 = k^2$. Thus straight lines passing through the origin will also be represented by the given equation; but this is merely a particular case.

(ii) The equation may be written in the form $\bar{x}^2 + \bar{y}^2 = k^2$; so that the corresponding curves have the centres of curvature at every point lying on the circle $x^2 + y^2 = k^2$. Evidently the doubly infinite series of circles with their centres on this circle satisfy the equation; but the complete analytic solution is troublesome and there is just a chance that other curves might be represented.

(iii) The equation may be written :

$$(mx - y)y_2 = (1 + my_1)(1 + y_1^2). \qquad \ldots (1)$$

Differentiating and re-arranging we have

$$(mx - y)y_3 = 3y_1y_2(1 + my_1). \qquad \ldots \qquad \ldots (2)$$

Dividing (2) by (1), $\frac{y_3}{y_2} = \frac{3y_1y_2}{1+y_1{}^2}$,

the primitive of which, as in the solution of (i) is

$$x^2 + y^2 + 2Ax + 2By + C = 0. \quad \ldots \quad \ldots (3)$$

The given equation may also be written in the form $\bar{y} = m\bar{x}$, $(\bar{x}, \bar{y})$ being the co-ordinates of the centre of curvature at any point. But for the circle represented by (3) we have $\bar{x} = -A$, $\bar{y} = -B$; hence we must have $B = Am$, so that the primitive becomes

$$x^2 + y^2 + 2Ax + 2Amy + C = 0. \quad \ldots \quad \ldots (4)$$

Thus the differential equation represents the doubly infinite series of circles with their centres on $y = mx$. When the vertex of any such circle is at a finite distance from the origin and the centre removes to an infinite distance, the circle becomes a straight line perpendicular to the given line; but this again is merely a particular case.

T. P. TRIVEDI.
K. J. SANJANA

Solutions.

Question 1215.

(F. H. V. GULASEKHARAM) :—Show that the trilinear equation to the pedal circle of a point (f, g, h) with respect to a triangle ABC may be expressed in the form :—

$$\begin{vmatrix} 0, & f/g + \cos C, & f/h + \cos B, & \alpha \\ g/f + \cos C, & 0, & g/h + \cos A, & \beta \\ h/f + \cos B, & h/g + \cos A, & 0, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} = 0.$$

Solution (1) by Martyn M. Thomas, (2) by the Proposer.

(1) Let D, E, F be the feet of the ⊥s from (f, g, h) on BC, CA, AB; and D′, E′, F′ the feet of the ⊥s from $(1/f, 1/g, 1/h)$ which is the isogonal conjugate to (f, g, h).

Then the pedal circle of (f, g, h) will pass through D, D′; E, E′; and F, F′. Its equation being homogeneous and of the second degree in α, β, γ can be expressed in the form

$$\begin{vmatrix} P & Q & R & \alpha \\ L & M & N & \beta \\ X & Y & Z & \gamma \\ \alpha & \beta & \gamma & 0 \end{vmatrix} = 0,$$ where P, Q, R; L, M, N; X, Y, Z are quantities to be determined ... ... (1)

Now the co-ordinates of D are $(0, \cos C + g/f, \cos B + h/f)$.

,,	,,	D′	$(0, \cos C + f/g, \cos B + f/h)$
,,	,,	E	$(\cos C + f/g, 0, \cos A + h/g)$
,,	,,	E′	$(\cos C + g/f, 0, \cos A + g/h)$
,,	,,	F	$(\cos B + f/h, \cos A + g/h, 0)$
,,	,,	F′	$(\cos B + h/f, \cos A + h/g, 0)$;

and obviously the given determinant is satisfied by the co-ordinates of D, D′, etc.

Thus the given determinant denotes a conic through DD′, EE′, FF′. The condition for a circle being satisfied the conic is a circle. Hence the result.

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(2) Now the equation

$$\begin{vmatrix} 0 & q_1 & r_1 & l\alpha \\ p_2 & 0 & r_2 & m\beta \\ p_3 & q_3 & 0 & n\gamma \\ \alpha & \beta & \gamma & 0 \end{vmatrix} = 0$$

represents a conic through the points $(0, q_1, r_1)$, $(p_2, 0, r_2)$ and $(p_3, q_3, 0)$.

Hence substituting the co-ordinates of the vertices of the pedal triangle of (f, g, h) for those of these three points and choosing $l : m : n$ so that the conic is a circle, the equation to the pedal circle assumes the form given in the question.

Question 1226.

(B. B. Bagi) :—A,B,C, D,E,F are six points on a conic. DE, DF meet BC in L_1, L_2; EF, ED meet CA in M_1, M_2; and FD, FE meet AB in N_1, N_2; then if AD, BE, CF meet in a point O, then L_1N_2, N_1M_2, M_1L_2 meet in the same point O.

Solution by S. Audinarayanan.

(1) Let DE, DF meet BC in L_1, L_2.
Join L_1 O and produce it to meet FE at N_2.
Join AN_2, DN_2, BA and FA.
Now,

$$\begin{aligned} N_2(D,A,X,O) &= (DAXO) \\ &= F(DAXO) \\ &= F(DAEC) \\ &= B(DAEC) = (DYEL_1). \end{aligned}$$

Since in the two ranges (DAXO) and $(DYEL_1)$, D is a common point and the two are equal, YA, EX and L_1O must be concurrent. Hence $BYAN_2$ lie on a straight line.

Thus L_1N_2 passes through O.

Similarly we can prove that N_1M_2, and M_1L_2 pass through the same point.

Thus the proposition is established.

(2) Project the conic into a circle with the projection of O for its centre.

AD, BE, CF become diameters of the circle centre O. Hence chords AB, DE, are ∥. Similarly AC ∥ DF and BC ∥ EF.

Thus AN_1DM_2 is a parallelogram and one of its diagonals N_1M_2 passes through the mid. pt. of its other diagonal AD.

Similarly N_2L_1, L_2M_1 pass through O.

Thus L_1N_2, N_1M_2, M_1L_2 are concurrent at O.

Hence they are concurrent in the original case also.

Question 1242.

(A. C. L. WILKINSON) :—Solve the equation

$$\log\left(x\frac{dy}{dx}+y\right)+\frac{2y}{x\dfrac{dy}{dx}+y}=a.$$

Solution by A. Mahalingam and N. M. Shah.

Putting $xy = v$, the equation transforms into

$$\log\frac{dv}{dx}+\frac{2v}{x\dfrac{dv}{dx}}=a,$$

or
$$2v = px\,(a-\log p) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

where $p=\dfrac{dv}{dx}$. Differentiating with respect to x, we have

$$2p = p\,(a-\log p)+x\,(a-\log p-1)\,\frac{dp}{dx}.$$

i. e. $$\frac{(a-\log p-1)}{(a-\log p-2)}\cdot d\,(\log p) = -\frac{dx}{x}$$

i. e. $$-\log x = \log A - 1\log p - \log\,(a-\log p-2)$$

∴ $$a-\log p-2 = Apx, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

A being an arbitrary constant.

Equation (1) gives $2y = p\,(a-\log p)$. $\qquad \ldots \qquad \ldots \qquad (3)$

The eliminant of p between (2) and (3) gives the solution,

$$2\,Axy = \left[\sqrt{1+2\,Axy}-1\right]\left[a-\log\left(\frac{\sqrt{1+2\,Axy}-1}{Ax}\right)\right]$$

Question 1243.

(A. C. L. WILKINSON) :—Solve the equation

$$(y - px)(3y - px) = p, \text{ where } p = \frac{dy}{dx} \text{ as usual.}$$

Show that the p-discriminant is a cusp locus and that the envelope of the tangents at the cusps consists of the two hyperbolas $4\sqrt{3}xy = \pm 1$.

Solution by A. Mahalingam and N. M. Shah.

By the application of the principle of Duality, the differential equation transforms into

$$Y(3Y - 2PX) = X$$

(*i.e.*)

$$3Y^2 - 2XY\frac{dY}{dX} = X.$$

$$\therefore \quad \frac{dY^2}{dX} - \frac{3}{X}Y^2 = -1.$$

Solving this linear equation, we have

$$2Y^2 = X + 2AX^3,$$

where A is an arbitrary constant.

Differentiating this with respect to x, and writing x for P, we have

$$6AX^2 + 1 - 4Yx = 0 \quad \ldots \quad \ldots \quad (1)$$

By the dual relations $X = p$, $Y = px - y$, this reduces to

$$6Ap^2 - 4px^2 + (1 + 4xy) = 0 \quad \ldots \quad (2)$$

Equation (2) together with the given equation,

$$p^2x^2 - p(1 + 4xy) + 3y^2 = 0$$

constitute the solution of the differential equation.

The elimination of p between these quadratic forms leads to the actual solution,

$$\{x^2(1+4xy) - 18Ay^2\}^2 = [(1+4xy)^2 - 12x^2y^2]\,[4x^4 - 6A(1+4xy)]$$

Again

$$(y - px)(3y - px) = p;$$

i. e.,

$$x^2p^2 - (4yx + 1)p + 3y^2 = 0. \quad \ldots \quad (1)$$

The p discriminant is

$$(4yx + 1)^2 - 12\,x^2y^2 = 0.$$

i. e., $$4\,(xy + 1)^2 - 3 = 0. \qquad \ldots \qquad \ldots \quad (2)$$

$\therefore$ $$xy + 1 = \pm \tfrac{1}{2}\sqrt{3}. \qquad \ldots \qquad \ldots \quad (3)$$

It can be easily shown that (2) is not a solution of (1); hence it represents the cusp-loci.

If (ξ, η) is a point on the primitive of (1) which is a cusp, then

$$\xi\eta = \alpha$$

where $\alpha = -1 \pm \frac{1}{2}\sqrt{3}$ and the tangent to the curve at (ξ, η) has its slope given by one of the equal roots of (1), the p-discriminant being equal to zero at (ξ, η).

$\therefore$ $$p = \frac{4\xi\eta + 1}{2\xi^2} = \frac{4\alpha + 1}{2\xi^2}$$

and the tangent is given by

$$y - \eta = \frac{4\alpha + 1}{2\xi^2}\,(x - \xi)$$

i.e. $$y - \frac{\alpha}{\xi} = \frac{4\alpha + 1}{2\xi^2}\,(x - \xi);$$

or $$2y.\xi^2 + (4\alpha + 1 - 2\alpha)\xi - (4\alpha + 1)\,x = 0.$$

The envelope is given by

$$(2\alpha + 1)^2 = -8\,(4\alpha + 1)\,xy.$$

Now $$\alpha = -1 \pm \frac{\sqrt{3}}{2}.$$

Substituting and simplifying

$$4\sqrt{3}xy = \pm 1.$$

Question 1244.

(S. RAJANARAYAÑAN) :—Sum the series

$$(a - 1)_0\, a_r - a_1 . a_{r-1} + (a + 1)_2\, a_{r-2} - \ldots\ldots\ldots (-1)^r (a + r - 1)_r\, a_0$$

where n_r denotes the number of combinations of n things r at a time.

Solution by I. Totadri Iyengar, N. M. Shah and J. B. Freeman.

The given series is the co-efficient of x^r in the product of $(1 + x)^a$ and $(1 + x)^{-a}$ as will be seen on expansion of these functions.

Hence it is equal to zero.

Question 1245.

(S. RAJANARAYAN) :—Find the value of the infinite series

$$S = S_1 + \frac{S_2}{2!} + \frac{S_3}{3!} + \text{..........}$$

where S_r denotes the sum of the rth powers of the first n natural numbers.

Solution by J. B. Freeman, I. Totadri Iyengar, A. Mahalingam, N. M. Shah, R. Jagannathan and G. V. Gurjar.

Arrange the terms in rows thus

$$\begin{aligned} S = &1 + 2 + 3 + \text{......................} + n \\ &+ \frac{1}{2!}\{1 + 2^2 + 3^2 + \text{......................} + n^2\} \\ &+ \frac{1}{3!}\{1 + 2^3 + 3^3 + \text{......................} + n^3\} \\ &+ \text{..} \\ &+ \frac{1}{s!}\{1 + 2^s + 3^s + \text{......................} + n^s\} \\ &+ \text{..} \end{aligned}$$

Summing by columns we see that

$$S = e + e^2 + e^3 + \text{..........................} + e^n - n.$$

$$= \frac{e^{n+1} - e}{e - 1} - n.$$

Question 1246.

(S. RAJANARAYANAN) :—Find the value of the expression

$$\sqrt{[a + \sqrt{\{ab + \sqrt{ab^3} + \sqrt{ab^7} + \sqrt{ab^{15}} + \dots\}}]}.$$

Solution by A. Mahalingam, J. B. Freeman, N. M. Shah, R. Jagannathan, I. Totadri Iyengar and G. V. Gurjar.

Let the expression be equal to x.

It is easily seen that

$$x = \sqrt{a + \sqrt{b}\,.\,x}.$$

Hence $$x^2 - a - \sqrt{b}\,x = 0,$$

whence $$x = \frac{\sqrt{b} \pm \sqrt{b + 4a}}{2}.$$

Question 1254.

(G. S. MAHAJANI) :—At a certain place A, on a certain day,

m_o = mean solar time of sidereal noon,

s_o = sidereal time of mean noon,

show that on that day the 'mean' and 'sidereal' times (m, s) at any other place B, are connected by the relation

$$\frac{m \pm L/15}{m_o} + \frac{s \pm L/15}{s_o} = 1,$$

where L° is the difference of the longitudes of B and A.

Solution by J. B. Mukherji and K. J. Sanjana.

At the given instant (m, s) at B the mean and sidereal times at A are $(m \pm \frac{1}{15} L)$ and $(s \pm \frac{1}{15} L)$ respectively.

Let k be the factor for transforming mean intervals into corresponding sidereal intervals.

As s_o is the sidereal time of mean noon at A, the sidereal time at A corresponding to mean time m at B is $s_o + k\left(m \pm \frac{1}{15} L\right)$; but this is

$$\left(s \pm \frac{1}{15} L.\right)$$

$$\therefore \quad s_o + k\left(m \pm \frac{1}{15} L\right) = s \pm \frac{1}{15} L.$$

By a similar reasoning, it is readily seen that

$$m_o + k^{-1}\left(s \pm \frac{1}{15} L\right) = m \pm \frac{1}{15} L.$$

Eliminating k between these equations, we have

$$\left(m \pm \frac{1}{15} L\right)\left(s \pm \frac{1}{15} L\right) = \left(s \pm \frac{1}{15} L - s_o\right)\left(m \pm \frac{1}{15} L - m_o\right);$$

or $$s_o\left(m \pm \frac{1}{15} L\right) + m_o\left(s \pm \frac{1}{15} L\right) = s_o m_o,$$

from which the required result follows.

Questions for Solution.

1285. (R. Vythynathaswamy):—If α, β, γ are the parameters of the inflexions in any system of parametric representation of the nodal cubic, shew that the points of contact of tangents from any point are parametrically determined by an equation of the form

$$\lambda(x-\beta)^2(x-\gamma)^2 + \mu(x-\gamma)^2(x-\alpha)^2 + v(x-\alpha)^2(x-\beta)^2 = 0.$$

Hence prove Mr. Youngman's result (Question 18078, *Educational Times*), that the conic passing through the node and the points of contact of tangents from any point on the line of flexes circumscribes the Δ of inflexional tangents.

1286. (B. B. Bagi):—Squares ABDE, ACFG, are described externally on the sides AB, AC, of ΔABC, L, M being their respective centres. H, K are the mid. pts. of BC, EQ and AK, AH meet them in P, Q; then shew that (1) L, P, M, H, K, Q lie on a circle of which HK, LM, are perpendicular diameters and that (2) four times the area of sq. HMKL = sq. AF + sq. AD + 4 Δ ABC.

1287. (B. B. Bagi):—If regular polygons $A_1A_2 \ldots A_m$; $B_1 B_2 B_3 \ldots B_n$ are inscribed in circles with centres a_1, a_2 and radii r_1, r_2 respectively, then shew that $\sum\limits_1^m \sum\limits_1^n A_x B_y^2 = mn\,(a_1 a_2^2 + r_1^2 + r_2^2)$.

[The circles are not necessarily in one plane.]

1288. (R. Gopalaswami):—A conic can be drawn through the centres of curvature of the max. inscribed ellipse at the mid.-points of the sides of the triangle to osculate the evolute of the ellipse and the point of contact is the centre of curvature of the fourth point of intersection of the max. ellipse with the Nine-Points circle.

1289. (K. J. Sanjana and T. P. Trivedi):—Solve the following differential equations and give their geometrical significance:—

$$\text{(i)} \quad x^2 + y^2 + \frac{(y - xy_1)(1 + y_1^2)}{y_2} = 0;$$

$$\text{(ii)} \quad y^2 - x^2 + \frac{(y + xy_1)(1 + y_1^2)}{y_2} = 0;$$

$$\text{(iii)} \quad nx^2 + y^2 + \frac{2(1 + y_1^2)(y - nxy_1)}{y_2} = 0.$$

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LIST OF JOURNALS RECEIVED BY THE SOCIETY.

1 Messenger of Mathematics
2 Quarterly Journal of Mathematics
3 Mathematical Gazette
4 The Annals of Mathematics
5 American Journal of Mathematics
6 Bulletin of the American Mathematical Society
7 Transactions of the American Mathematical Society
8 Monthly Notices of the Royal Astronomical Society
9 Proceedings of the Royal Society of London
10 The Philosophical Magazine and Journal of Science
11 Astrophysical Journal
12 Crelle's Journal
13 L'intermediaere des Mathematiciens
14 Mathematische Annalen
15 Philosophical Transactions of the Royal Society of London
16 Acta Mathematica
17 Popular Astronomy
18 Proceedings of the Edinburgh Mathematical Society
19 Proceedings of the London Mathematical Society
20 Mathematics Teacher
21 Bulletin of the Calcutta Mathematical Society
22 The Tohoku Mathematical Journal
23 Nature
24 The American Mathematical Monthly
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